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Control of chaotic systems with uncertain parameters and stochastic disturbance by LMPC

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***Abstract*—For the chaotic systems with uncertain parameters and stochastic disturbance, in order to satisfy some optimal performance index when chaos control is achieved, the Lyapunov-based model predictive control (LMPC) is introduced. The LMPC scheme is concerned with an auxiliary controller which is constructed in advance. Based on the auxiliary controller and stochastic stability theory, it is shown that the chaotic systems with uncertain parameters and stochastic disturbance are practical stable. With the help of the auxiliary controller, the stability of LMPC can be guaranteed as well as some optimality property. As an example, the unified chaotic system with uncertain parameter and stochastic disturbance is considered and simulation results show the effectiveness of the proposed method.**

***Index Terms*—chaos, predictive control, uncertain parameters, stochastic disturbance, optimization.**

I. INTRODUCTION

Chaos is a complex nonlinear phenomenon that the behavior of dynamical systems is highly sensitive to initial conditions, which is also referred to as the butterfly effect. It exists in many practical fields such as biology, economics, engineering, finance, physics, oscillating chemical reactions, fluid and so on. Because of the sensitiveness on initial conditions, chaos is unpredictable in the long time and may be undesired in some applications. In order to suppress this undesirable behavior, Ott, Grebogi and Yorke first presented a kind of controlling chaos method, which is so called OGY control method [1]. Since then, chaos control has been a hot issue and many techniques have been presented [2], for example, delayed feedback control method [3], adaptive control method [4], linear and nonlinear control methods [5]–[7], active control method [8], etc.

In real systems or experimental situations, it is difficult to obtain the exact model. Instead, people usually only know the approximate system model, and uncertain or/and stochastic disturbance exists inevitably. It is important and necessary to discuss the control problem when the systems include uncertain or/and stochastic disturbance. On the other hand, in many practical control fields such as economics, engineering, finance and so on, the control problem is often connected with some optimal performance index for example saving costs (money, energy), gaining the most profits, and even some constraints. Obviously, the above question is concerned with the optimal control problem of the systems with uncertain or/and stochastic disturbance. In many application fields, model predictive control [9] is adopted widely when the control problem includes optimal performance, constraints, uncertainty, etc. MPC is a kind of optimal control technique, but where it differs from the traditional optimal control method is that it solves the standard optimal control problem on-line in a finite horizon, rather than determining off-line a feedback law. Also, when the on-line solution is obtained, MPC typically sends out the first control action to be implemented, and repeats the calculation at the next instant. The advantages of MPC are that it can handle constraints, owns the ability of prediction especially when there exists time delay, and to some extent overcomes the effect of uncertainty on the systems.

Motivated by the above discussions, in this paper, we consider the optimization and control problem of chaotic systems with uncertain parameters and stochastic disturbance by the Lyapunov-based model predictive control [10]–[16]. By using a previously designed Lyapunov controller, the stability is discussed and proved by stochastic Lyapunov stability theory. As a typical example, the unified chaotic system with the uncertain parameter and stochastic disturbance is considered and simulation results show the effectiveness of the proposed method.

II. CONTROL OF CHAOTIC SYSTEMS WITH UNCERTAIN PARAMETERS AND STOCHASTIC DISTURBANCE BY LMPC

In many MPC formulations there often exist two important issues that how to guarantee the closed-loop stability and initial conditions starting from where the control is feasible. In order to solve these problems, the Lyapunov-based model predictive control(LMPC) is presented [10]–[16]. The idea of LMPC is that a Lyapunov-based controller $h(x)$ is designed previously, and the closed-loop stability and the optimization are based on the controller $h(x)$. With the help of the controller $h(x)$, the stability of LMPC can be inherited and the region of optimization feasibility can be explicitly characterized.

Let us consider the control problem of the following chaotic systems

$$\dot{x}(t) = f_0(x(t)) + f_1(x(t))\theta(x(t)) + l(x(t))\xi(t) + g(x(t))u(t) \tag{1}$$

where $f_0(0) = f_1(0) = l(0) = g(0) = 0$, namely the origin is the equilibrium point; $\theta(x(t))$ is the uncertain parameter and there exists $\theta_b > 0$ such that $\|\theta(x(t))\| \leq \theta_b$; $\xi(t)$ denotes the standard Gaussian white noise which can be expressed as the formal derivative of Wiener process $w(t)$; $u(t)$ is the controller. We suppose that all the states of the system are available, $f_0(x), f_1(x), \theta(x), l(x), g(x)$ are continuous functions, and compared to the originally deterministic system $f_1(x(t))\theta(x(t)) + l(x(t))\xi(t)$ can be viewed as the small disturbance.

For a candidate Lyapunov function $V(x)$, the following feedback control law $u(t) = h(x(t))$ can be constructed

$$h(x) = \begin{cases} 0, & if\ L_gV(x) = 0 \\ -\frac{\omega+\sqrt{\omega^2+(L_gV(L_gV)^T)^2}}{L_gV(L_gV)^T}(L_gV)^T, & else \end{cases} \tag{2}$$

where $\omega = L_{f_0}V(x) + ||L_{f_1}V(x)||\theta_b + \frac{1}{2}Trace\left\{l^T\frac{\partial V^2}{\partial x^2}l\right\} + \rho V(x)$, $\rho > 0$, $L_{f_i}V(x) = \frac{\partial V(x)}{\partial x}f_i(x)(i = 0,1)$ and $L_gV(x) = \frac{\partial V(x)}{\partial x}g(x)$. It can be investigated that if $V(x)$ is a control Lyapunov function, then $h(x)$ is optimal in some sense [16]–[18].

The LMPC can be designed as follows

$$\min_{u(\tau)\in S(\Delta)} \int_{t_k}^{t_k+T} (\hat{x}^T(\tau)Q\hat{x}(\tau) + u^T(\tau)Ru(\tau))d\tau \tag{3}$$

$$\dot{\hat{x}}(\tau) = f_0(\hat{x}(\tau)) + g(\hat{x}(\tau))u(\tau) \tag{4}$$

$$\hat{x}(t_k) = x(t_k) \tag{5}$$

$$\frac{\partial V(x(t_k))}{\partial x}g(x(t_k))u(t_k) \le \frac{\partial V(x(t_k))}{\partial x}g(x(t_k))h(x(t_k)) \tag{6}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period Δ, which means that the controller is applied in a sample-and-hold fashion. Q, R are positive definite weight matrices. LMPC is unnecessary to use a terminal penalty term, but needs an auxiliary Lyapunov-based control law $h(x)$ that gives the contractive constrains of the Lyapunov-based model predictive controller. With the help of the controller $h(x)$, the initial feasibility of the optimization is satisfied automatically. In the following section, we will prove the stability of the system (1) under the control action $u(t_k)$, which further implies that the optimization is consecutively feasible.

In order to make the theory analysis easily, we give the following assumption:

Assumption 1:

(1) $\|\Psi(y, \theta(y), u) - \Psi(x, \theta(x), u)\| \le \eta_\Psi \|y - x\|$, $\eta_\Psi > 0$. where $\Psi(x, \theta(x), u) = L_{f_0}V(x) + L_{f_1}V(x)\theta(x) + \frac{1}{2}Trace\left\{l(x)^T\frac{\partial V^2}{\partial x^2}l(x)\right\} + L_gV(x)u(t)$.

(2) $\|F(x, \theta(x), u)\| \le M_1$,$\|l(x(t))\| \le M_2$, $(M_1 > 0, M_2 > 0)$. where $F(x, \theta(x), u) = f_0(x) + f_1(x)\theta(x) + g(x)u(t)$.

Remark 1: For the chaotic system, its attractor is a bounded compact set. When the control input is added and not too large, usually the boundedness can be kept. Based on the properties that the functions $f_0(x), f_1(x), \theta(x), l(x), g(x)$ are continuous, then we can conclude the assumption (2) holds. Finally,if the function $\Psi(x, \theta(x), u)$ is further differential in x for each u, we can obtain $\Psi(x, \theta(x), u)$ satisfies the Lipschitz property in the assumption(1). Thus, through the above discussion we know that Assumption 1 is not difficult to be satisfied.

Lemma 1: If $\dot{x}(t) = F(x(t), \theta(x(t)), u(t_k)) + l(x(t))\xi(t), t \in [t_k, t_{k+1})$, where $F(x(t), \theta(x(t)), u(t_k)) = f_0(x(t)) + f_1(x(t))\theta(x(t)) + g(x(t))u(t_k))$, then there exists a constant $\gamma > 0$ such that the following inequality holds $E\|x(t) - x(t_k)\| \le \gamma\sqrt{\Delta}$.

Proof.

$$\dot{x}(t) = F(x(t), \theta(x(t)), u(t_k)) + l(x(t))\xi(t),\ t \in [t_k, t_{k+1})$$

Integrating the above formula, we get $x(t) - x(t_k) = \int_{t_k}^t F(x(\tau), \theta(x(\tau)), u(t_k))d\tau + \int_{t_k}^t l(x(\tau))dw$

Applying the inequality $\|a + b\|^2 \le 2\|a\|^2 + 2\|b\|^2$, then

$$\begin{aligned} &E\left(\|x(t) - x(t_k)\|^2\right) \\ &= E\left\|\int_{t_k}^t F(x(\tau), \theta(x(\tau)), u(t_k))d\tau + \int_{t_k}^t l(x(\tau))dw\right\|^2 \\ &\le 2E\left\|\int_{t_k}^t F(x(\tau), \theta(x(\tau)), u(t_k))d\tau\right\|^2 \\ &\quad +2E\left\|\int_{t_k}^t l(x(\tau))dw\right\|^2 \end{aligned}$$

Using Cauchy-Schwarz inequality and Assumption 1, we can obtain

$$\begin{aligned} &E\left(\left\|\int_{t_k}^t F(x(\tau), \theta(x(\tau)), u(t_k))d\tau\right\|^2\right) \\ &\le (t - t_k)E\left(\int_{t_k}^t \|F(x(\tau), \theta(x(\tau)), u(t_k))\|^2 d\tau\right) \\ &= (t - t_k)\int_{t_k}^t E\left(\|F(x(\tau), \theta(x(\tau)), u(t_k))\|^2\right)d\tau \\ &\le M_1^2(t - t_k)^2 \le M_1^2\Delta^2 \end{aligned}$$

$$\begin{aligned} &E\left(\left\|\int_{t_k}^t l(x(\tau))dw\right\|^2\right) = E\left(\int_{t_k}^t \|l(x(\tau))\|^2 d\tau\right) \\ &\le M_2^2(t - t_k) \le M_2^2\Delta \end{aligned}$$

Thus

$$E\left(\|x(t) - x(t_k)\|^2\right) \le 2(M_1^2\Delta + M_2^2)\Delta$$

If we let $\gamma^2 = 2(M_1^2\Delta + M_2^2)$, then by Jensen's inequality we get

$$\begin{aligned} &E\|x(t) - x(t_k)\| = E\sqrt{\|x(t) - x(t_k)\|^2} \\ &\le \sqrt{E\left(\|x(t) - x(t_k)\|^2\right)} \le \gamma\sqrt{\Delta}. \end{aligned}$$

Theorem 1: Consider the trajectory $x(t)$ of the system (1) under the control law $u(t)$, which satisfies the conditions of Assumption 1 and is implemented in a sample-and-hold fashion:

$$\dot{x}(t) = f_0(x(t)) + f_1(x(t))\theta(x(t)) + l(x(t))dw + g(x(t))u(t_k) \tag{7}$$

where $t \in [t_k, t_{k+1})$ and $t_k = t_0 + k\Delta, k = 1, 2, ...$

Then the origin of the system (1) is practically stable in some mean sense.

Proof. The time derivative of the Lyapunov function $V(x)$ along the trajectory $x(t)$ of the system (1) in $t \in [t_k, t_{k+1})$ is given by

$$\dot{V}(x(t)) = \Psi(x(t), \theta(x(t)), u(t_k)) + \frac{\partial V(x(t))}{\partial x} l(x(t)) dw$$

Adding and subtracting $\Psi(x(t_k), \theta(x(t_k)), u(t_k))$, and taking into account Assumption 1 and the constraint (6), we obtain

$$\begin{aligned}
&\dot{V}(x(t)) \leq -\rho V(x(t_k)) + \Psi(x(t), \theta(x(t)), u(t_k)) \\
&-\Psi(x(t_k), \theta(x(t_k)), u(t_k)) + \tfrac{\partial V(x(t))}{\partial x} l(x(t)) dw \\
&\leq -\rho V(x(t_k)) + \eta_\Psi ||x(t) - x(t_k)|| + \tfrac{\partial V(x(t))}{\partial x} l(x(t)) dw
\end{aligned}$$

Taking the expectation of the above inequality and using Lemma 1, it leads to

$$\begin{aligned}
&E\dot{V}(x(t)) \\
&\leq -\rho EV(x(t_k)) + \eta_\Psi E||x(t) - x(t_k)|| \\
&\leq -\rho EV(x(t_k)) + \gamma \eta_\phi \sqrt{\Delta}
\end{aligned}$$

When the right side of the above inequality is less than zero, we know that the value of $EV(x(t))$ will decrease. Therefore, if we take $r_1 > r_2 > 0$ and a small constant $\varepsilon > 0$ such that $r_2 = (\varepsilon + \gamma\eta_\Psi\sqrt{\Delta})/\rho$, $\Omega_{r_1} = \{x : EV(x(t)) \leq r_1\}$, $\Omega_{r_2} = \{x : EV(x(t)) \leq r_2\}$, then for $x(t) \in \Omega_{r_1}/\Omega_{r_2}$, the state will converge to Ω_{r_2} in some mean sense. If we further take $r_{\min} < r_1$ and $r_{\min} = \max\limits_{\Delta_1 \in [0,\Delta]} \{EV(x(t + \Delta_1)) : EV(x(t)) \leq r_2\}$, then once the state converge to $\Omega_{r_2} \subseteq \Omega_{r_{\min}}$, it will remains inside $\Omega_{r_{\min}}$ for all times. That is to say $\lim\limits_{t\to\infty} supEV(x(t)) \leq r_{\min}$.

III. EXAMPLE

Let us consider the control problem of the unified chaotic system with the uncertain parameter θ and stochastic disturbance $l(x)\xi(t)$.The system (8) is called the unified chaotic system when the right of (8) only includes $f(x(t), \theta)$ and $\theta \in [0, 1]$. As shown in [19], if $\theta \in [0, 0.8)$, it is called the generalized Lorenz chaotic system; if $\theta = 0.8$, it is called Lu chaotic system; if $\theta \in (0.8, 1]$, it becomes the generalized Chen system. Therefore, the parameter θ plays an important role in the unified chaotic system, and it's natural to discuss the control problem when the parameter θ is uncertain [20], [21]. Furthermore, stochastic disturbance exists inevitably in practice. It is necessary to discuss the control problem of the unified chaotic system with uncertain parameter and stochastic disturbance.

$$\dot{x}(t) = f(x(t), \theta) + l(x(t))\xi(t) + Bu(t) \tag{8}$$

where

$$f(x, \theta) = \begin{pmatrix} (25\theta + 10)(x_2 - x_1) \\ (28 - 35\theta)x_1 + (29\theta - 1)x_2 - x_1x_3 \\ -(8 + \theta)x_3/3 + x_1x_2 \end{pmatrix} \tag{9}$$

$$l(x(t))dw = \begin{pmatrix} \sigma_1 x_1 & 0 & 0 \\ 0 & \sigma_2 x_2 & 0 \\ 0 & 0 & \sigma_3 x_3 \end{pmatrix} \begin{pmatrix} dw_1 \\ dw_2 \\ dw_3 \end{pmatrix} \tag{10}$$

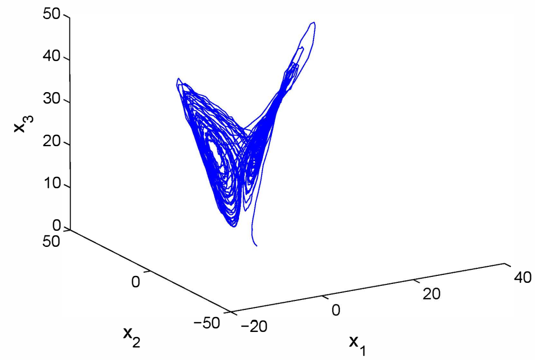

Fig. 1. Phase portrait of chaotic attractor in (x_1, x_2, x_3) space.

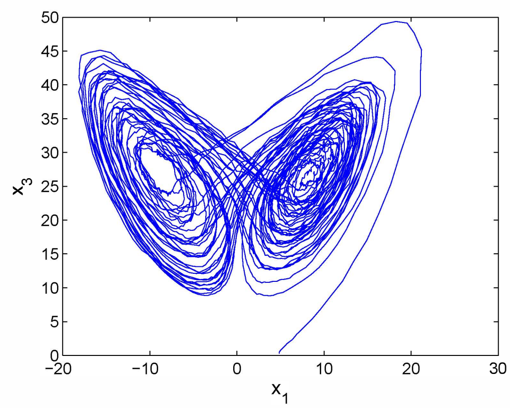

Fig. 2. Projective portrait of chaotic attractor in (x_1, x_3) plane.

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{11}$$

In this example, we take $l(x)$ and B as (10) and (11), respectively. Choosing $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, we get $\Psi(x, \theta, u) = a(x, \theta) + x_2 u$, where

$$\begin{aligned}
&a(x, \theta) = (x_1, x_2, x_3)f(x, \theta) + \tfrac{1}{2}(\sigma_1^2x_1^2 + \sigma_2^2x_2^2 + \sigma_3^2x_3^2) \\
&= (38 - 10\theta)x_1x_2 + (29\theta - 1)x_2^2 - (10 + 25\theta)x_1^2 - \tfrac{8+\theta}{3}x_3^2 \\
&+\tfrac{1}{2}(\sigma_1^2x_1^2 + \sigma_2^2x_2^2 + \sigma_3^2x_3^2)
\end{aligned}$$

when $x_2 = 0, x \neq 0$, and $\theta \in [0, 1], \sigma_1 < 2\sqrt{5}, \sigma_3 < 4/\sqrt{3}$,

$$\alpha(x, \theta) = -(25\theta + 10 - \frac{1}{2}\sigma_1^2)x_1^2 - (\frac{8 + \theta}{3} - \frac{1}{2}\sigma_3^2)x_3^2 < 0$$

Thus, $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ is a control lyapunov function [17], and the auxiliary controller $h(x)$ is taken as (2).

In numerical simulations, we take the initial value $x(0) = (2, -1, 1)^T$ and the parameters $\sigma_1 = \sigma_2 = \sigma_3 = 0.1, \theta(x) = 0.1\,|\sin(x_1)|\,, \rho = 0.001, Q = I, R = 1.0, \Delta = 0.02, T = 100\Delta$, to investigate the proposed method. Figs. 1-2 displays the phase portrait of the uncertain and stochastic chaotic attractor when the system (8) is without control input. When

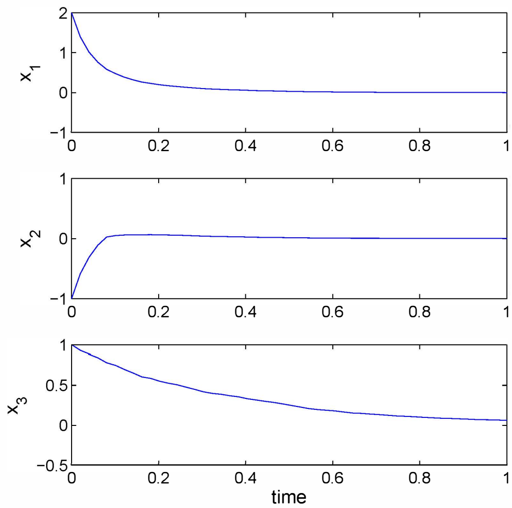

Fig. 3. The evolution of system states under LMPC.

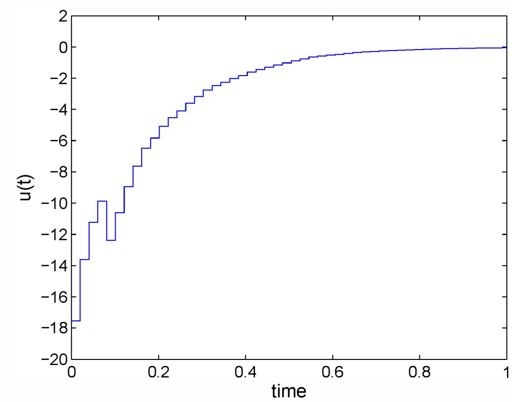

Fig. 4. The varying curve of control input by LMPC.

the control law of LMPC is applied to the chaotic system, from Figs. 3-4 one can see that the state trajectories of the chaotic system and the control action will approach to the neighborhoods of zero points as time increases. These results investigate the effectiveness of the proposed method for the chaotic systems with uncertain parameters and stochastic disturbance.

IV. Conclusion

Chaos is usually undesirable in real systems, therefore many methods are proposed to control it. In this paper, for the chaotic systems with uncertain parameters and stochastic disturbance, we discussed the chaotic control problem by using the Lyapunov-based model predictive control (LMPC). Through introducing the auxiliary control law $h(x)$, the stability of LMPC is discussed and proved by stochastic Lyapunov stability theory. Compared to other chaotic control schemes, the advantage of the proposed method is that the optimality can be considered and guaranteed as well as the close-loop stability. Simulation results show the effectiveness of the proposed method.

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